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On weak approximation in algebraic groups and related varieties defined by systems of forms

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Abstract

It is well-known that weak approximation holds for a large class of semisimple groups over global fields, including those which are simply connected or adjoint. Earlier Kneser suggested the investigation of weak approximation in algebraic groups over any field of definition and Platonov gave examples of simply connected groups of type ^1A which do not have this property. Thus conjecturally adjoint groups satisfy weak approximation over arbitrary fields of definition. Here we prove the validity of weak approximation for many adjoint semisimple groups over arbitrary fields of definition and also for varieties, defined by a system of forms, which are closely related to adjoint groups.

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0. Introduction

Let k be a field, W a set of valuations of k, X an irreducible k-variety. We say that X has *weak approximation over k with respect to W* if for any finite subset S of W, via the diagonal embedding, $X(k)$ is dense in the product $\prod_{v\in S} X(k_v)$ in the product topology of the v-adic topologies induced from k_v, the completion of k at v. X is *rational* (resp. *stably rational*) *over k* if the function field $k(X)$ of X is a pure transcendental extension (resp. subfield of such an extension) of k. The problem of deciding whether a given variety X has weak approximation property or is rational or stably rational over k has attracted the attention of many people and plays an important role in the geometry and arithmetic of the variety X (cf. e.g. [5,25] and references there).

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If X is the underlying variety of an algebraic group G, then Kneser [11, 12] was the first to show that weak approximation holds over global fields for simply connected groups of classical type, and then Platonov (cf. e.g. [22]) extended this to all simply connected groups. In [9] Harder proved the weak approximation property over global fields for a large class of semisimple groups, including the adjoint ones, and later there are some other proofs and refinements (see [13, 25]). Earlier in [11] Kneser suggested studying weak approximation in algebraic groups over arbitrary fields and in [21] Platonov gave examples to show that this problem has a negative answer for simply connected almost simple groups of type ^{1}A. (Later on Monastyrnyi and Yanchevski gave similar examples for simply connected groups of types ^{2}A and D, cf. [19] and the reference there.) There is a conjecture (due to Platonov), which states that if G is an adjoint semisimple algebraic group over k, then G is k-rational as a k-variety (see [22, p. 465]). In particular it always satisfies weak approximation over k.

In our earlier (unpublished) preprint [28] (resp. in [30]) we have discussed and proved (resp. announced some results on) the validity of weak approximation property in many adjoint almost simple k-groups over an arbitrary field k. In particular, in our papers [28–31] we first pointed out that for adjoint groups G of classical types C and D, a factor group of the group of similarity factors of the corresponding (skew-) hermitian form is a *rational obstruction* of the group G. Quite recently (after the first version of this paper had been submitted) A. Merkurjev [18] gave examples of groups of classical types, for which the above conjecture about the rationality is not true. The main ingredient used by Merkurjev is the study of the group of similarity factors which we earlier used in [28–31]. In particular, the Scharlau Norm Principle (see [31]) was also used.

Since there is a growing interest in this problem, and the weak approximation problem for adjoint groups is still open, we would like to give here full proofs of the results obtained in [28, 30] with some refinements and clarifications. After we give some general results about weak approximation in Section 1, we prove weak approximation for adjoint groups of classical (resp. exceptional) types in Section 2 (resp. Section 3). In Section 4 we discuss the weak approximation property for varieties defined by quadratic forms which are closely related to adjoint groups of type C and D. In Section 5 we prove a general theorem about *integral approximation* over global fields, by applying the methods developed in previous sections.

Notation and convention. Throughout this paper, unless otherwise stated, G is a linear algebraic group defined over a field k of characteristic 0 or p sufficiently large, V is the set of all non-equivalent valuations of k, and $\mathrm{H}^1(\cdot,\cdot)$ is the 1-Galois cohomology set in the usual sense. $\mathbf{G}_m$ denotes the multiplicative k-group, $Z_G(A)$ denotes the centralizer of the set A in a group G. We assume that all quadratic or (skew-) hermitian forms are non-degenerate. The notation of indices of almost simple groups is the same as in [33]. A k-root of G is called *isotropic* if it corresponds to a vertex of the index of G which itself is circled; otherwise, it is called *quasi-isotropic*. For a finite set S of valuations of k, we set $G(S) := \prod_{v\in S} G(k_v)$. Denote by $Cl_S(G(k))$ the closure of $G(k)$ in the product topology in $G(S)$, and by $A(S,G) = G(S)/Cl_S(G(k))$ the obstruction to weak approximation in S.

Let D be an associative finite-dimensional division algebra over k and f a (skew-) hermitian form over a finite-dimensional right D-vector space V with values in D. Denote by $\mathrm{GU}(f,D)$ (resp. $\mathrm{U}(f,D)$, $\mathrm{SU}(f,D)$) the similarity (resp. unitary, special unitary) group of the form f, and by $\mathrm{GU}(f)$ (resp. $\mathrm{U}(f)$, $\mathrm{SU}(f)$) the corresponding k-group. Then $\mathrm{GU}(f,D) = \mathrm{GU}(f)(k)$, etc. is the group of k-points of $\mathrm{GU}(f)$, etc.

We say the form f is of *type* A (resp. C, D) if the k-group $\mathrm{SU}(f)$ is of type A (resp. C, D) in the sense of algebraic groups. If f is of type D, we denote by $\mathrm{GU}^+(f)$ the connected component of the group $\mathrm{GU}(f)$. We set $\mathrm{G}(f) = m(\mathrm{GU}(f)(k))$, $\mathrm{G}^+(f) = m(\mathrm{GU}^+(f)(k))$, where m is the projection $m : \mathrm{GU}(f) \to \mathbf{G}_m$ mapping each element $g \in \mathrm{GU}(f)$ to its similarity factor $m(g)$. We call $\mathrm{G}(f)$ (resp. $\mathrm{G}^+(f)$) *the group of similarity factors* (resp. *special similarity factors*) of f. In [7] Dieudonné mentioned that in general case, the determination of the group $\mathrm{G}(f)$ is an open problem, the solution of which depends on the arithmetic of the field k. Here we relate the study of the group $\mathrm{G}(f)$ to the study of weak approximation in adjoint algebraic groups and algebraic varieties defined by a system of forms. We will define an arithmetic invariant for forms f of type C and D, which turns out to be also a *weak approximation obstruction* of the adjoint group of $\mathrm{SU}(f)$ over any field k of characteristic $\neq 2$.

For a division k-algebra D and a (skew-) hermitian form f with values in D, we denote by $D^*(f)$ the set of non-zero values of the form f in D, $D_v = D \otimes k_v$, $f_v = f \otimes k_v$. For such f we say that $\mathrm{G}(f)$ *satisfies weak approximation over k with respect to V* if for any finite set $S \subset V$, $\mathrm{G}^+(f)$ is dense in $\mathrm{G}^+(f)(S) = \prod_{v\in S} \mathrm{G}^+(f_v)$ in the product topology-induced from $\prod_{v\in S} k_v$. If this property holds for any form f of type C or D we say that the field k *satisfies condition* (S). From [29, 31] it follows that any global field of char. $\neq 2$ satisfies the condition (S). In this paper we show that if the k-rank of G is large enough and the degree of the division algebra related with G is small then weak approximation holds for G. More precisely, one of main results of this paper is the following (see Section 3).

Theorem. *Let G be an adjoint k-group.*

(a) *If every simple k-factor of G is one of the following types*: ${}^1\mathrm{A}_n$ $(n \geq 1)$; ${}^2\mathrm{A}_{2n}$, $n \geq 1$; ${}^2\mathrm{A}^{(d)}_{2rd+1,r}$, *$d$ odd*; B_n $(n \geq 2)$; $\mathrm{C}^{(d)}_{n,r}$ $(\max(n-rd,d) \leq 2 \leq n)$; $\mathrm{D}^{(d)}_{n,r}$ $(\max(n-rd,d) \leq 2)$; $\mathrm{E}_{6,r}$ $(r \geq 2)$; $\mathrm{E}_{7,r}$ $(r \geq 3)$; $\mathrm{E}_{8,r}$ $(r \geq 4)$; $\mathrm{F}_{4,r}$ $(r \geq 1)$; G_2, *then G has the weak approximation property over k.*

(b) *If every simple k-factor of G is either one of the types described above or of type $\mathrm{C}^{(d)}_{n,r}$ or $\mathrm{D}^{(d)}_{n,r}$ and k satisfies the condition* (S) *then G has the weak approximation property over k.*

(c) *If every simple k-factor of G is either one of the types described in* (a) *or of type $\mathrm{E}_{n,r}$ ($r \geq 1$, the types E_7^{78}, E_8^{78}, E_8^{133} are excluded) and if over k any k-group of type D_n with $4 \leq n \leq 7$ has weak approximation, then G does also.*

In general, since it is hard to prove the rationality or irrationality of a given group or variety, it is useful to have general criteria for these properties. Recently, we have found some general criteria for stable rationality of almost simple algebraic groups, (see [32]). From this one can also deduce the weak approximation property, since we have (strict) implications

rationality $\Longrightarrow$ stable rationality $\Longrightarrow$ weak approximation.

1. General remarks

In this section we give some general facts about weak approximation in linear algebraic groups over an arbitrary field k. Let

$$1 \to A \to B \xrightarrow{f} C \to 1$$

be an exact sequence of k-groups, with A central in B and f separable. Let S be a finite subset of the set V' of valuations of rank 1 from V, f_v the induced map $B(k_v) \to C(k_v)$ and f_S the product of f_v's. We have the following commutative diagram with exact rows and all arrows are natural morphisms.

$$\begin{tikzcd} B(k) \arrow[r] \arrow[d] & C(k) \arrow[r, "\delta"] \arrow[d] & \mathrm{H}^1(k,A) \arrow[d, "\gamma"] \\ B(S) \arrow[r] & C(S) \arrow[r, "\delta_S"] & \prod_{v\in S} \mathrm{H}^1(k_v,A) \end{tikzcd}$$

Lemma 1.1. *If the sets* $\mathrm{H}^1(k,A)$ *and* $\mathrm{H}^1(k_v,A)$ *are endowed with discrete topologies then* δ *and* δ_S *are continuous mappings.*

Proof. Since f is separable, the map f_S is open by the Implicit Function Theorem (see e.g. [10]), hence $f_S(B(S))$ is open in $C(S)$. We have only to show that δ_S is continuous at the identity element of $C(S)$, since δ_S is a homomorphism of topological groups. But this follows from the fact that $\delta^{-1}(1) = f_S(B(S))$ is open, where 1 denotes the neutral element of $\prod_{v\in S} \mathrm{H}^1(k_v,A)$. $\square$

Lemma 1.2. *With notation as above, suppose that* B *satisfies weak approximation with respect to* S. *Then there is a one-to-one correspondence between* $A(S,C)$ *and the factor set* $\delta_S(C(S))/\gamma(\delta(C(k)))$. *In particular,* C *has weak approximation with respect to* S *if and only if* $\delta_S(C(S)) = \gamma(\delta(C(k)))$.

Proof. From the above diagram we get the following commutative diagram:

$$\begin{tikzcd} B(k) \arrow[r] \arrow[d] & C(k) \arrow[r, "\delta"] \arrow[d] & \delta(C(k)) \arrow[r] \arrow[d] & 1 \\ B(S) \arrow[r] & C(S) \arrow[r, "\delta_S"] & \delta_S(C(S)) \arrow[r] & 1 \end{tikzcd}$$

Assuming weak approximation for B with respect to S, by using the fact that $f_S(B(S))$ is an open hence closed subgroup of $C(S)$, we deduce

$$f_S(B(S)) = f_S(Cl_S(B(k))) = Cl_S(f_S(B(k))) \subset Cl_S(C(k)). \tag{1}$$

Therefore,

$$f_S(B(S))C(k) \subset Cl_S(C(k)). \tag{2}$$

Since $f_S(B(S))$ is an invariant open subgroup of $C(S)$, the left-hand side in (2) is an open subgroup of $C(S)$ containing $C(k)$. Hence $f_S(B(S))C(k) = Cl_S(C(k))$. Since $f_S(B(S))$ is the kernel of δ_S, we have

$$A(S,C) = C(S)/f_S(B(S))C(k) = \delta_S(C(S))/\delta_S(C(k)).$$

Since $\delta_S(C(k)) = \gamma(\delta(C(k)))$, the lemma follows. □

Remark. In Lemma 1.2 we need mainly the continuity of δ and δ_S, which can also be proved to hold if we endow $\mathrm{H}^1(k,A), \mathrm{H}^1(k_v,A)$ with some good separable topologies. Moreover, with the help of Theorem 1.4 and Lemma 1.7 below one can prove Lemma 1.2 just by assuming that the valuations under consideration are henselian, but we do not need this fact in the sequel. The idea of Lemma 1.2 is due to Kneser [11], who first gave a cohomological criteria for weak approximation in algebraic groups.

From Lemma 1.2 we deduce the following:

Corollary 1.3. *In the situation of Lemma* 1.2, *suppose that there is a k-subgroup D of B containing A such that* $\mathrm{H}^1(k_v,D) = 0$ *for all* $v \in S$, *and that D and* $E = D/A$ *satisfy weak approximation property with respect to S. Then C does also and* δ_S *is surjective.*

Proof. We identify $E = D/A$ with a subgroup of C. We have

$$\gamma(\delta(E(k))) \subset \gamma(\delta(C(k))) \subset \delta_S(C(S)) \subset \prod_{v\in S} \mathrm{H}^1(k_v,A).$$

From the assumption we have $\delta_S(C(S)) = \gamma(\delta(C(k)))$ and δ_S is surjective. □

Remark. The Corollary 1.3 still holds if we drop the condition that all valuations considered have rank 1, which is what we need in the sequel.

The following theorem is the main result of this section. It shows that the measure of obstruction to the weak approximation property of a connected reductive k-group and that of its parabolic k-subgroup and its Levi subgroup are the same. It is also an important tool in proving our main theorem. For standard notations, terminologies and results from theory of reductive groups we refer to [1] and [2].

Theorem 1.4. *Let Q be a connected reductive k-group, P a parabolic k-subgroup of Q,* $P = MR_u(P)$ *a Levi decomposition of P and S a finite subset of V. Then we have the following bijections*:

$$A(S,Q) \leftrightarrow A(S,P) \leftrightarrow A(S,M).$$

In particular, Q satisfies weak approximation with respect to S if and only if P (and so M) does.

Proof. Let $U = R_u(P)$. First we prove the second bijection. Since, $P = MU$ is a semidirect product, we have

$$P(k) = M(k)U(k), \quad P(k_v) = M(k_v)U(k_v), \quad \forall v \in S.$$

Hence,

$$Cl_S(P(k)) = Cl_S(M(k))Cl_S(U(k)) = Cl_S(M(k))\left(\prod_{v \in S} U(k_v)\right),$$

since U is a k-split unipotent group and thereby also satisfies weak approximation over k. Therefore,

$$A(S,P) = M(S)U(S)/Cl_S(M(k))Cl_S(U(k)) = A(S,M).$$

Let P_0 be a minimal k-parabolic subgroup of Q contained in P, $U_0 = R_u(P_0)$, $P_0 = M_0U_0$, where M_0 is a Levi subgroup of P_0 such that $M_0 \subset M$. By P_0^- and U_0^- we denote the parabolic k-subgroup of Q opposite to P_0 and its unipotent radical, respectively. The notation P^-, U^- have similar meaning. In particular, $P_0^- \subset P^-$. Then for any extension K of k we have

$$Q(K) = U_0(k)U_0^-(K)P_0(K) \tag{3}$$

(see [1, Proposition 6.25]). The proof of the Lemma 6.28 of [1] (which is crucial for the proof of (3)) shows that (3) also holds where K is a finite product of fields which are extensions of k. In particular, we have

$$Q(S) = U_0(k)U_0(S)P_0(S). \tag{4}$$

We claim that for any such K as above

$$Q(K) = U(K)U^-(K)P(K) = U(K)U^-(K)M(K)U(K), \tag{5}$$

and in particular

$$Q(S) = U(S)U^-(S)M(S)U(S). \tag{6}$$

Indeed, by the choice of P and P_0 we have $U_0(K)M_0(K) \subset U(K)M(K)$. Hence from (3) we have

$$\begin{aligned} Q(K) &= U_0(k)U_0^-(K)P_0(K) \\ &= U_0(k)U_0^-(K)M_0(K)U_0(K) \\ &= U_0(k)M_0(K)U_0^-(K)U_0(K) \\ &\subset U(K)M(K)(M_0(K)U_0^-(K))U_0(K) \end{aligned}$$

$$\begin{aligned}&\subset U(K)M(K)(M(K)U^-(K))U_0(K)\\&\subset U(K)U^-(K)M(K)M_0(K)U_0(K)\\&\subset U(K)U^-(K)M(K)U(K).\end{aligned}$$

Now we consider the natural map

$$P(S)/Cl_S(P(k)) \xrightarrow{\beta} Q(S)/Cl_S(Q(k)),$$

where $pCl_S(P(k)) \mapsto pCl_S(Q(k)), p \in P(S)$. We claim that

(a) *β is surjective.*

In fact, by (5) we have

$$\begin{aligned}Q(k_v) &= U(k_v)U^-(k_v)P(k_v)\\&\subset Cl_S(U(k))Cl_S(U^-(k))P(k_v)\\&\subset Cl_S(Q(k))P(k_v),\end{aligned}$$

since U, U^- have the weak approximation property. From this we derive

$$Q(S) = Cl_S(Q(k))P(S) = P(S)Cl_S(Q(k)),$$

which means that β is surjective. (One can use the same argument by applying the equality (6).)

(b) *β is injective.*

To show that β is injective, we need to check that $Cl_S(Q(k)) \cap P(S) = Cl_S(P(k))$. It is easy to see that this amounts to checking $Cl_S(Q(k)) \cap M(S) = Cl_S(M(k))$. Since the v-adic topology is Hausdorff, from (5) and from the weak approximation property of U we have

$$Cl_S(Q(k)) = U(S)U^-(S)U(S)Cl_S(M(k)). \tag{7}$$

Now let $z \in Cl_S(Q(k)) \cap M(S)$, $z = uu_1u'm$, where $u, u' \in U(S)$, $u_1 \in U^-(S)$, $m \in Cl_S(M(k))$ (by (7)). Hence,

$$u_1 = u^{-1}zm^{-1}u'^{-1} \in M(S)U(S) \cap U^-(S) = \{1\}.$$

Thus $z = m$ as required.

The last statement follows directly from the previous one, or one can apply [11, Section 2.1]. $\square$

As consequence we derive the following:

Corollary 1.5. *Let Q be a connected reductive k-group, quasi-split over k and let T be a maximal k-torus of a Borel k-subgroup B of Q. Then we have a bijection*

$$A(S,Q) \leftrightarrow A(S,B) \leftrightarrow A(S,T).$$

Remark. From results of [25] it follows that if k is a number field, the corollary also holds for any reductive k-group with special covering (in the terminology of [25]) without the assumption of being quasi-split.

The next two lemmas are due to Tits.

Lemma 1.6 (cf. [27, p. 35], [34, Lemme, p. 89]). *Let G be a simple adjoint k-group, S' a maximal k-split torus of G and $Z_G(S')$ the centralizer of S' in G. Then for some central anisotropic k-torus T' of $Z_G(S')$, $S'T'$ is the center of $Z_G(S')$.*

Proof. It is an immediate consequence of [1, Proposition 6.8]. Another argument can be given as follows. We fix a maximal k-torus T of G containing S'. Denote by Φ the root system of G with respect to T. Since G is an adjoint group, the roots of Φ generate the character group $X(T)$, hence the center T_1 of $Z_G(S')$ is given by

$$T_1 = \bigcap_{\alpha \in \Delta_0} \operatorname{Ker} \alpha = \bigcap_{\alpha \in \Phi, \alpha_{|S'} = 1} \operatorname{Ker} \alpha = \bigcap_{\alpha \in \mathbf{Z}\Delta_0} \operatorname{Ker} \alpha,$$

where Δ_0 denotes the subset of all simple roots of Φ which are trivial on S'. Since

$$X(T) = \mathbf{Z}\Delta = \mathbf{Z}\Delta_0 \oplus \mathbf{Z}(\Delta \backslash \Delta_0),$$

i.e. $\mathbf{Z}\Delta_0$ is a direct factor of $X(T)$, and T_1 is the annihilator of $\mathbf{Z}\Delta_0$ in T, it follows that T_1 is a torus, containing S'. Since S' is maximal k-split, there is an anisotropic k-torus T' such that $T_1 = S'T'$. □

Lemma 1.7 (cf. [35, pp. 505–515]). *Let G be an almost simple k-group and S' a maximal k-split torus. Then the connected center of $Z_G(S')$ is a k-induced torus, thus is cohomologically trivial.*

Proof. Let S_0 be the connected center of $Z_G(S')$. Let Δ_0, Δ be as above. Consider the k-index of G (see [33, Section 2.3]). Then it is not hard to see that there is a basis of the character group $X(S_0)$ of S_0 on which $Gal(k_s/k)$ acts by permutations. □

The following two results will be used frequently.

Lemma 1.8 (Ono [20]). *Let T be a central k-torus of a connected reductive k-group G. If $\mathrm{H}^1(K, T) = 0$ for all extensions K of k (e.g. if T is an induced k-torus) then there is a k-section $G/T \to G$. In particular, G is k-birationally equivalent to the product $(G/T) \times T$.*

Lemma 1.9 (Đoković and Thang [8]). *Let ${}_k\Delta$ be a basis of the relative k-root system and S_θ $(\theta \subset {}_k\Delta)$ be a standard k-split torus of a connected reductive k-group G. Let $Z_G(S_\theta) = S_\theta T_0 H$ (almost direct product), where T_0 is a k-torus, H a semisimple k-subgroup of G. Then the Tits index of H is obtained from that of G by removing all vertices not belonging to the preimage $\tilde{\theta}$ of $\theta \cup \{0\}$ under the restriction map $\Delta \to {}_k\Delta \cup \{0\}$. Moreover, T_0 is anisotropic and $ST_0 = (Z_T(H))^0$.*

2. Weak approximation for groups of classical types

In this section we prove the weak approximation property for many adjoint semisimple groups over arbitrary fields of definition. We assume all adjoint groups considered here to be simple. The idea is first to prove the weak approximation property for groups of classical type and then reduce the exceptional cases to the classical. Further we will freely use the notation of [33]. First we need to recall the following important

Theorem 2.1 (Voskresenski and Klyachko [37]). *Let G be an adjoint k-group of type A_{2n}. Then G is rational as a k-variety.*

Theorem 2.2. *Let G be a simple adjoint k-group of type A_n or B_n. Then G has weak approximation property over k if one of the following holds:*

(a) *G is of inner form;*

(b) *G is of outer type ${}^2A_{2n}, n \geq 1$ or ${}^2A^{(d)}_{2rd+1,r}$.*

(c) *G is of outer type ${}^2A_n$ which becomes of inner type over an extension k'/k such that the norm $N_{k'/k}$ is surjective. (This type will be denoted by ${}^2A^*$.)*

Proof. The case B_n is well-known by virtue of the Cayley transformation in any characteristic (see [7]). For groups of type A_n the only non-trivial case is the case ${}^2A^{(d)}_{2rd+1,r}$. In fact, the adjoint groups of type ${}^1A_n$ are known to be rational. The even rank case follows from Theorem 2.1. However, for the convenience of the reader we briefly recall (well-known) arguments for the case ${}^1A_n$.

(a) G is of type ${}^1A_n$. (Using the weak approximation in unitary groups (see [11]) the following argument also works for the case c.) Denote by $\tilde{G}$ the k-universal covering of G. Then $\tilde{G}(k) = SL_r(D)$, where D is a division algebra of finite dimension over its center k. Let ${}_kGL$ be a k-form of the group GL_{n+1} such that ${}_kGL(k) = GL_r(D)$. It is well-known that $G(k) = PGL_r(D)$. To see this we may use the exact sequence

$$1 \to \mathbf{G}_m \to {}_kGL \to G \to 1$$

of k-groups and derive from this the exact sequence

$$1 \to \mathbf{G}_m(k) \to GL_r(D) \to G(k) \to H^1(k, \mathbf{G}_m) = 0.$$

From this we have the following surjective homomorphisms:

$$\pi : GL_r(D) \to G(k), \quad \pi_v : GL_r(D \otimes k_v) \to G(k_v)$$

for all valuations v. Since the group ${}_kGL$ has the weak approximation property (i.e. $GL_r(D)$ is dense in $\prod_{v \in S} GL_r(D \otimes k_v)$ for any finite set S of valuations), the same holds for G.

(b) G is of type ${}^2A^{(d)}_{2rd+1,r}$. Let S' be a maximal k-split torus of G and T' a central k-torus of $Z_G(S')$ as in Lemma 1.6. Then by using [33], Lemmas 1.6 and 1.9 one sees that $Z_G(S')/S'T'$ is rational over k. Since $S'T'$ is an induced k-torus (Lemma 1.7) it is k-rational. From Lemma 1.8 we deduce that $Z_G(S')$ also is rational, hence G is also by Bruhat decomposition. In particular, G has weak approximation.

(c) Let GL' be a k-form of GL such that $\mathrm{GL}'(k) = \mathrm{U}(\Phi, D)$, where Φ is a non-degenerate hermitian form with values in a division k-algebra D with center k', and $\tilde{G} = \mathrm{SU}(\Phi)$ is the universal cover of G. Let T be the center of GL', which is a one-dimensional anisotropic k-torus $\mathrm{R}^{(1)}_{k'/k}(\mathbf{G}_m)$. We have the following exact sequence of k-groups:

$$1 \to T \to \mathrm{GL}' \xrightarrow{\pi} G \to 1,$$

where π is surjective on k-points due to the assumption. Hence for any finite set I of valuations of k, π induces a surjective map $\pi_I : \mathrm{GL}'(I) \to G(I)$. Since GL' is rational (see e.g. [11]) it also has weak approximation. Therefore, G has weak approximation property over k. □

Now we consider groups of type C_n and D_n. We have

Proposition 2.3. *Let G be an adjoint isotropic k-group of type ${}^3\mathrm{D}_4$, ${}^6\mathrm{D}_4$ or type $\mathrm{D}_{4,2}$ (non-trialitarian type). Then as a k-variety, G is k-rational.*

Proof. Let S be a maximal k-split torus of G. If the k-rank of G is 2, then the assertion is clear, due to the following facts:

(1) $Z_G(S)/S$ is a reductive k-group of rank 2 and any k-torus of dimension 2 is k-rational by a theorem of Voskresenski [36];

(2) By a result of Chevalley [3, Proposition 4], the function field of G over k is isomorphic to the function field of a generic maximal torus over a purely transcendental extension of k.

Assume that k-rank of G is 1. In this case from Lemma 1.9 it follows that $Z_G(S)/S$ is k-isomorphic to a group H which is a direct product of adjoint groups of type A_1. If G is of inner type over a cubic extension F of k, then $H \simeq \mathrm{R}_{F/k}(A)$, where A is a k-group of type A_1, hence H is k-rational. If G is of inner type over a Galois extension F with the group S_3, then G is of type ${}^6\mathrm{D}_4$ and becomes of type ${}^3\mathrm{D}_4$ over a quadratic extension K of k, and $H \simeq \mathrm{R}_{F/k}(A)$, where A is a F-group of type A_1. In all cases H is k-rational. By Lemma 1.8, $Z_G(S)$ is k-birationally equivalent to the direct product $S \times (Z_G(S)/S)$ hence is also k-rational. □

By making use of results in Section 1 and the above results for groups of type A we can prove the following result. In what follows, (P) stands for one of the following properties: weak approximation, rationality or stable rationality.

Theorem 2.4. *Assume that for a given field k there is a natural number m such that any adjoint k-group of (classical) type C_p (resp. D_p) has the property* (P) *over k with $p \leq m$. Then any adjoint k-group of type $\mathrm{C}^{(d)}_{n,r}$ (resp. $\mathrm{D}^{(d)}_{n,r}$) does also, provided $n - rd \leq m$ and $d \leq 2$.*

Proof. Let G be a group of type $\mathrm{C}_{n,r}^{(d)}$ (resp. $\mathrm{D}_{n,r}^{(d)}$) and let S be a maximal k-split torus of G. If G is of type D we may assume that it is a non-trialitarian type. If the index of G does not contain quasi-isotropic k-roots then (by Lemmas 1.6 and 1.9) $Z_G(S)/S$ is k-isomorphic to a direct product of adjoint k-groups of types A_{d-1} and C_{n-rd} (resp. D_{n-rd}). From the assumption and Lemma 1.8 it follows that $Z_G(S)/S$ has the property (P), hence G does also (see Theorem 1.4 or use the Bruhat decomposition). If the Tits index of G contains a quasi-isotropic k-root then by [33] $d = 1$ or 2. If $d = 1$, then G is quasi-split over k (see [33]). It is well-known and easy to prove that any almost simple quasi-split k-group is k-rational by making use of the Bruhat decomposition. If $d = 2$, then $n - rd = 1$ and by Lemma 1.9, $Z_G(S) = S \cdot R \cdot G_1$, where G_1 is a product of k-groups of type A_1, and R is a one-dimensional k-anisotropic torus. Note that SR is the connected center of $Z_G(S)$ which is rational over k and contains the center of G_1, the factor group $G' = Z_G(S)/SR$ is a direct product of k-groups of type A_1, hence also is rational over k. Hence from the exact sequence

$$1 \to SR \to Z_G(S) \to G' \to 1$$

we conclude that $Z_G(S)$ also is rational over k (by Lemma 1.8) and so is G. □

The following corollary shows that if the k-rank is large and the degree of the division involved is small, then the group satisfies weak approximation.

Corollary 2.5. *Let G be an adjoint group of type $\mathrm{C}_{n,r}^{(d)}$ (resp. $\mathrm{D}_{n,r}^{(d)}$) over a field k. Then G is rational over k if* $\max(n - rd, d) \leq 2$.

Proof. It follows from the above proposition and from the fact that adjoint groups of types C_2 (resp. $\mathrm{D}_1, \mathrm{D}_2$) as we have shown above, are rational over k. □

Let G be a simple adjoint k-group of type C or D (non-exceptional type), $\tilde{G}$ the unitary k-covering of G and f a (skew-) hermitian form corresponding to G (i.e. $\tilde{G} = \mathrm{SU}(f)$). We set

$$\mathrm{G}^+(S, f) = \prod_{v \in S} \mathrm{G}^+(f_v)/Cl_S(\mathrm{G}^+(f))$$

(see notation in the introduction). We may define in similar way the set $\mathrm{G}(S, f)$. We assume that f takes values in a division k-algebra D of degree d. The case when G is of type D is especially important because in many cases one can reduce the problem of verifying the property (P) above to the case of a group of type D. In general, the weak approximation problem for arbitrary adjoint groups G of type C and D over any field k is still open, so it is useful to have a simple (arithmetic) obstruction (to weak approximation, hence also to stable rationality or rationality) of the group G in this case. We have

Proposition 2.6. *With the above notation let f_0 be the maximal anisotropic subform of f. Then there are bijections of factor sets*

$$A(S, G) \leftrightarrow \mathrm{G}^+(S, f) \leftrightarrow \mathrm{G}^+(S, f_0).$$

Proof. We have $\mathrm{GU}^+(f) = \mathrm{SU}(f)\mathbf{G}_m$, where $\mathrm{SU}(f) \cap \mathbf{G}_m = \{\pm 1\}$. By Lemma 1.8 we have the following birational k-isomorphism:

$$\mathrm{GU}^+(f) \simeq \mathbf{G}_m \times G.$$

Therefore, we have a bijection of factor sets

$$A(S, \mathrm{GU}^+(f)) \leftrightarrow A(S, G).$$

Now we consider the following exact sequence of k-groups:

$$1 \to \mathrm{SU}(f) \to \mathrm{GU}^+(f) \to \mathbf{G}_m \to 1.$$

From this we derive the following exact sequences on k-points and k_v-points:

$$1 \to \mathrm{SU}(f)(k) \to \mathrm{GU}^+(f)(k) \to \mathrm{G}^+(f) \to 1,$$

$$1 \to \mathrm{SU}(f)(k_v) \to \mathrm{GU}^+(f)(k_v) \to \mathrm{G}^+(f_v) \to 1.$$

It is well-known that the group $\mathrm{SU}(f)$ is k-rational hence it has weak approximation over k, i.e. $A(S, \mathrm{SU}(f))$ is trivial for any S. Therefore, we have a bijection of factor sets,

$$A(S, \mathrm{GU}^+(f)) \leftrightarrow \mathrm{G}^+(S, f)).$$

Now we claim that $\mathrm{G}^+(f) = \mathrm{G}^+(f_0)$ (and similarly $\mathrm{G}^+(f_v) = \mathrm{G}^+(f_{0,v})$ for all v). Recall that

$$\mathrm{G}^+(f) = \{m(g) \in \mathrm{G}(f) : Nrd_{D/k}(g) = m(g)^{nd/2}\}$$

(the degree d of D is a power of 2). Therefore, if we denote by f_1 the hyperbolic part of f, then $\mathrm{G}(f_1) = \mathrm{G}^+(f_1) = k^*$ as is not hard to see. Thus $\mathrm{G}^+(f) = \mathrm{G}^+(f_0)$ and the proposition follows. □

Remark. This proposition shows that to verify the weak approximation property for G, we may assume that G is anisotropic (or isotropic) over k. It is interesting to note that if $d = 1$ or $d > 2$ then $\mathrm{G}(f) = \mathrm{G}^+(f)$ for any form of type D (see [29]). The following shows that the arithmetic invariant $\mathrm{G}^+(S, f)$ of the form f is in fact a *birational obstruction* of G.

Corollary 2.7. *With the above notation, if G is stably k-rational then $\mathrm{G}^+(S, f))$ is trivial for any S. Hence $\mathrm{G}^+(S, f)$ is a birational obstruction of G.*

If f is a quadratic form over k, we say that f is *round* if either f is hyperbolic or it is anisotropic and $D^*(f) = \mathrm{G}(f)$. It is well-known that any Pfister form is round but it is worth noticing that the converse is not true (see [16]). We have

Corollary 2.8. *Let f be a round form over k. Then the adjoint group of $\mathrm{SO}(f)$ satisfies weak approximation over k.*

Proof. It is easy to see that for any valuation v we have $D^*(f_v) = \mathrm{G}(f_v) = \mathrm{G}^+(f_v)$. Since $D^*(f)$ has weak approximation property, $\mathrm{G}(f) = \mathrm{G}^+(f)$ does also. □

Remarks. (1) If k is a global field, then we can use the computations of groups $\mathrm{G}^+(f)$ for forms f of type C or D in [29, 31] to prove that any adjoint group of type C or D over k satisfies weak approximation over k.

(2) Let k be a field of characteristic $\neq 2$. From the Harder–Milnor Theorem (cf. [14, 26]) it follows that if V_0 denotes the set of all valuations of a rational function field $K = k(t)$ which are trivial on k, and f is quadratic form over K, then $\mathrm{G}(f) = \bigcap_{v \in V_0} \mathrm{G}(f_v)$. Therefore the following statement (which is equivalent to the weak approximation property for the adjoint group of $\mathrm{SO}(f)$) seems to be true:

(S) *For any finite set S of V_0, $\mathrm{G}(S, f))$ is trivial.*

We prove the validity of (S) under some restrictions on the residue class fields of k_v.

Proposition 2.9. *Let K, V_0, f be as above. Assume that for almost all valuations $v \in V_0$ the residue class field $\kappa(v)$ of K_v is quadratically closed. Then* (S) *holds, i.e., the adjoint K-group G of* $\mathrm{SO}(f)$ *satisfies the weak approximation property over K.*

Proof. Let S_0 be any finite set of valuations in V_0 and S_1 the (finite) set of all valuations $v \in V_0$, where $\kappa(v)$ is not quadratically closed. Set $S = S_0 \cup S_1$. It suffices to prove that $A(S, \mathrm{G}(f))$ is trivial. For $v \in S$ let $x_v \in \mathrm{G}(f_v)$. By weak approximation in the field, there is an element $x \in K^*$ which is sufficiently close to every x_v. Since $\mathrm{G}(f_v)$ is an open subgroup in K_v, we may assume that $x \in \mathrm{G}(f_v)$ for all $v \in S$. For v not in S every binary quadratic form over K_v is universal. It is clear that we may assume f has dimension $2m$. Then from [30, p. 792], it follows that $x \in \mathrm{G}(f_v)$ for all v not in S. Thus, $x \in \mathrm{G}(f_v)$ for all $v \in V_0$, i.e. $xf \simeq f$ over K_v for all v. By the Harder–Milnor Theorem, $xf \simeq f$ over K, i.e. $x \in \mathrm{G}(f)$. Therefore for any given tuple $(x_v)_{v \in S}$ there is $x \in \mathrm{G}(f)$ sufficiently close to x_v in the v-adic topology. The proposition follows. □

We assume the same notation of Proposition 2.8. Denote by W the set of all valuations of K which are *non-trivial on k*. By trivial extension of valuations $v \in V$ to valuations on K, i.e., $v(t) = 0$, we may regard V as a subset of W. Denote $R = V_0 \cup W$. We have the following exact sequence of K-groups:

$$1 \to \mu_2 \to \mathrm{SO}(f) \xrightarrow{\pi} G \to 1,$$

where $\mu_2 = \{\pm 1\}$, and the following comutative diagram with exact rows:

$$\begin{tikzcd}
\mathrm{SO}(f)(K) \arrow[r, "\pi"] \arrow[d, "i"] & G(K) \arrow[r, "\delta"] \arrow[d, "j"] & H^1(K, \mu_2) \arrow[r, "\beta"] \arrow[d, "\phi"] & H^1(K, \mathrm{SO}(f)) \arrow[d, "\gamma"] \\
\prod_v \mathrm{SO}(f)(K_v) \arrow[r, "\pi'"] & \prod_v G(K_v) \arrow[r, "\delta'"] & \prod_v H^1(K_v, \mu_2) \arrow[r, "\alpha"] & \prod_v H^1(K_v, \mathrm{SO}(f))
\end{tikzcd} \qquad (*)$$

where v runs over R. Note that for $v \in W$, the completion K_v of K contains the field $k_v(t)$, where v is considered as a valuation of k by restricting v to k, (cf. N. Bourbaki, *Commutative Algebra*, Ch. VI, Section 10). We make the following condition.

Condition (H). *There are natural Hausdorff topologies on the cohomology sets* $H^1(k, \mathrm{SO}(f))$, $H^1(K_v, \mathrm{SO}(f))$ *and the product*

$$\prod_v H^1(K_v, \mathrm{SO}(f))$$

such that all maps in the above diagram are continuous with respect to corresponding topologies and the natural topologies on $\mathrm{SO}(f)(K_v)$, $G(K_v)$ *and* $H^1(K_v, \mu_2)$ *induced from the v-adic topology of the field K.*

This condition is fulfilled, for example when k is a global field. Then we have:

Proposition 2.10. *With the above notation and condition* (H), *the group G satisfies weak approximation with respect to R, as a K-group. In particular, since f is defined over k, G satisfies weak approximation with respect to V as a k-group.*

Proof. We use here some ideas due to Kneser [11]. First we claim that ϕ has a dense image in the product topology. For, consider the following exact sequence of K-groups:

$$1 \to \mu_2 \to \mathrm{SL}_2 \to \mathrm{PSL}_2 \to 1,$$

where, as usual, SL_2 denotes the K-split special linear group, etc. Since the group SL_2 has trivial cohomology and PSL_2 has the weak approximation property over K (because PSL_2 is a K-split group), from the following commutative diagram with exact rows:

$$\begin{tikzcd}
\mathrm{PSL}_2(K) \arrow[r] \arrow[d] & H^1(K, \mu_2) \arrow[r] \arrow[d, "\phi"] & \{0\} \\
\prod_v \mathrm{PSL}_2(K_v) \arrow[r] & \prod_v H^1(K_v, \mu_2) \arrow[r] & \{0\}
\end{tikzcd}$$

the claim follows.

In the diagram $(*)$ the map γ has trivial kernel by the Harder–Milnor Theorem (see [14,26]) and by a twisting argument, it is also injective. Let g be any element of $\prod_v G(K_v)$. Then $\delta'(g)$ is belong to the closure of the image of $\mathrm{H}^1(K, \mu_2)$ under ϕ, by the claim above. There is a generalised sequence $(x_a)_{a \in I}$ of elements of $\mathrm{H}^1(K, \mu_2)$ such that $\phi(x_a) \to \delta'(g)$. Here I denotes a directed set of indices. By the assumption on continuity, $\gamma(\beta(x_a)) = \alpha(\phi(x_a))$ converges to $\alpha(\delta'(g)) = 1$. Hence $\beta(x_a) \to 1$ (in $\mathrm{H}^1(K, \mathrm{SO}(f))$), since γ is injective and continuous. Therefore given any open neighborhood U of $\delta(G(K))$ there is $\alpha_0 \in I$ such that $x_a \in U$ provided $a > a_0$. We denote this fact by the symbol $x_a \sim \delta(G(K))$, i.e., x_a is "very close" to $\delta(G(K))$ for large a. Due to the continuity of ϕ we have $\phi(x_a) \sim \phi(\delta(G(K)) = \delta'(G(K))$. Meanwhile $\phi(x_a) \to \delta'(g)$, and since the topology is Hausdorff, we must have $\delta'(g) \in Cl(\phi(\delta(G(K)))) =$

$Cl(\delta'(G(K)))$, where Cl denotes the closure in the topology of $\prod_v \mathrm{H}^1(k_v, \mu_2)$. Hence

$$g \in \delta'^{-1}(Cl(\delta'(G(K)))) = Cl(G(K))\pi'\left(\prod_v \mathrm{SO}(f)(K_v)\right)$$
$$= Cl(G(K))\pi'(Cl(\mathrm{SO}(f)(K))) \subset Cl(G(K)),$$

where Cl denotes the closure in the corresponding topology. This means that an arbitrary element $g \in \prod_v G(K_v)$ is in $Cl(G(K))$, hence we are done.

The second assertion follows from the first one. For, we have

$$k(t) \subset k_v(t) \subset k(t)_v,$$

where $k(t)_v$ denotes the completion of $k(t)$ in v. For any finite set $S \subset V$, we set

$$G(S) = \prod_{v \in S} G(k_v)$$

and assume that $(x_v)_{v\in S} \in G(S)$ is given. Since G satisfies weak approximation over $k(t)$ with respect to R, there is an element $x(t) \in G(k(t))$ which is sufficiently close to $(x_v)_{v\in S}$ in the product topology of $\prod_{v\in S} G(k(t)_v)$. Note that this product topology induces the ordinary product topology on $\prod_{v\in S} G(k_v)$. Since S is finite, we can choose a suitable specialization $t \to t_0$, where $t_0 \in k$, such that $x(t_0) \in G(k)$, which is still sufficiently close to $(x_v)_{v\in S}$ in the product topology. □

3. Weak approximation for groups of exceptional types. Main theorem

In this section we show that many adjoint k-groups satisfy weak approximation if their k-rank is positive.

Theorem 3.1. *Let G be an isotropic adjoint k-group of type* $\mathrm{E}_{6,r}$, $r \geq 2$. *Then G satisfies weak approximation over k.*

Proof. The proof is similar to that in the previous section and is based on the consideration of Tits' indices [33] and Lemma 1.9. We may assume that G is neither k-split nor k-quasi-split. In all cases let S be a maximal k-split torus of G.

(a) G is of type ${}^1\mathrm{E}^{28}_{6,2}$. It is well-known that $Z_G(S) = S \cdot G_1$, where G_1 is the group $\mathrm{Spin}(f)$ with f a norm-form of a Cayley algebra defined over k. Since S contains the center of G_1 by Lemma 1.6, $Z_G(S)/S \simeq \mathrm{PSO}(f)$ which satisfies weak approximation by Corollary 2.8. Hence G does also, by Proposition 1.4.

(b) G is of type ${}^1\mathrm{E}^{16}_{6,2}$ or ${}^2\mathrm{E}^{16''}_{6,2}$. Then by Lemma 1.6, $Z_G(S)/S$ is a direct product of two adjoint k-groups of type A_2, hence is rational over k. In particular, G is also rational over k.

(c) G is of type ${}^2\mathrm{E}^{16'}_{6,2}$ or ${}^2\mathrm{E}^{2}_{6,4}$. We consider only the first case since the second is similar. Then we have $Z_G(S) = SRG_1$, where G_1 is a k-group of type ${}^2\mathrm{A}_3 \simeq {}^2\mathrm{D}_3$.

We claim that the simply connected covering $\tilde{G}_1$ of G_1 is isomorphic to the group $\mathrm{Spin}(f)$, where f is an anisotropic nondegenerate 6-dimensional quadratic form over k. Here $\tilde{G}_1 = \mathrm{SU}(\Phi)$, where Φ is a non-degenerate hermitian form associated with a division algebra D over a quadratic extension k' of k. Indeed, there is a quadratic extension $L = k(\sqrt{a})$ $(a \in k)$ of k such that G is L-split. Therefore, $D \otimes L$ is split and D is a quaternion algebra over k'. If $\sqrt{a} \notin k'$ then G is of inner type over k' and D is not split, which is impossible. Therefore $L = k'$ and $D = k'$. Then it is well-known that there is a 6-dimensional quadratic form f such that $\mathrm{SU}(\Phi, k') \simeq \mathrm{Spin}(f, k)$ and the claim is proved.

By Theorem 1.4 and Lemma 1.6, it is sufficient to prove the weak approximation property for $Ad(G_1)$, the adjoint group of G_1. Since $f \otimes L$ is hyperbolic, it is well-known that $f \simeq \langle 1, -a\rangle \cdot \langle b, c, d\rangle$, where $b, c, d \in k^*$. From [29, p. 792] it follows that $\mathrm{G}(f) = N_{L/k}(L^*)$. (In fact it is true for any quadratic form which is a multiple of $\langle 1, -a\rangle$.) Therefore it is easy to see that the weak approximation holds for $\mathrm{G}(f)$, i.e., also for G as required. $\square$

The same method of proof gives us the following:

Proposition 3.2. *Let G be an adjoint k-group of type* E_7 *of k-rank ≥ 3. Then G satisfies weak approximation over k.*

Proposition 3.3. *Let G be a k-group of type* E_8, F_4 *or* G_2. *Assume the k-rank of G is > 3 (resp. > 0) if G is of type* E_8 *(resp.* F_4*). Then G satisfies weak appproximation over k.*

Proof. We consider, for example, the case $\mathrm{F}_{4,1}$. The other cases are proved by using similar ideas and/or methods used above. Let S be a maximal k-split torus of G. Then again by Lemma 1.9, $Z_G(S) = SG_1$, where G_1 is a simply connected k-group of type B_3. S necessarily contains the center of G_1 by Lemma 1.6, therefore as above $Z_G(S)/S \simeq \mathrm{SO}(f)$, where f is a quadratic form over k in 7 variables. Since $\mathrm{SO}(f)$ is k-rational, we are done. $\square$

Proposition 3.4. *Let G be a simple adjoint k-group.*

(a) *Let G be one of the following types* ${}^2\mathrm{E}_{6,1}^{29}$, $\mathrm{E}_{7,1}^{66}$, $\mathrm{E}_{7,1}^{48}$, $\mathrm{E}_{7,2}^{31}$, $\mathrm{E}_{8,1}^{91}$, $\mathrm{E}_{8,2}^{66}$. *Assume that any adjoint k-group of type* D_n *with $4 \leq n \leq 7$ satisfies weak appproximation over k. Then the same holds for G.*

(b) *Assume that G is of type* $\mathrm{E}_{7,1}^{78}$ *(resp.* $\mathrm{E}_{8,2}^{78}$ *or* $\mathrm{E}_{8,1}^{133}$*). Then G satisfies weak approximation over k if its universal covering or any adjoint k-group of type* E_6 *(resp. any adjoint k-group of type* $\mathrm{E}_{8,1}^{78}$ *or* E_7*) does.*

Proof. The only non-trivial case is the case $\mathrm{E}_{7,1}^{78}$ of (b). Let $\tilde{G}$ be the universal k-covering of G and $\pi : \tilde{G} \to G$ the canonical projection. Assume that $\tilde{G}$ satisfies weak approximation. If π is inseparable, then the assertion is trivial. Otherwise let S be a maximal k-split torus of $\tilde{G}$. One can check easily that S contains the center of $\tilde{G}$. Then Corollary 1.3 finishes the proof. □

Finally we obtain the following main theorem.

Theorem 3.5. *Let G be an adjoint k-group.*

(a) *If every simple k-factor of G is of one of the following types:* ${}^1\mathrm{A}_n$ $(n \geq 1)$; ${}^2\mathrm{A}_{2n}$, $n \geq 1$; ${}^2\mathrm{A}^{(d)}_{2rd+1,r}$, d odd; ${}^2\mathrm{A}^*$; B_n $(n \geq 2)$; $\mathrm{C}^{(d)}_{n,r}$, $\max(n - rd, d) \leq 2 \leq n$; $\mathrm{D}^{(d)}_{n,r}$, $\max(n - rd, d) \leq 2$; $\mathrm{E}_{6,r}$ $(r \geq 2)$; $\mathrm{E}_{7,r}$ $(r \geq 3)$; $\mathrm{E}_{8,r}$ $(r \geq 4)$; $\mathrm{F}_{4,r}$ $(r \geq 1)$; G_2 *then G satisfies weak approximation over k.*

(b) *If every simple k-factor of G is either as above or of type* C *or* D *and k satisfies the condition* (S), *then G satisfies weak approximation over k.*

(c) *If every simple k-factor of G is one of the types in* a) *or of type* $\mathrm{E}_{n,r}$ $(r \geq 1$, *the types* E^{78}_7, E^{78}_8, E^{133}_8 *are excluded) and if over k any adjoint k-group of type* D_n *with* $4 \leq n \leq 7$ *satisfies weak approximation then G does also.*

Proof. It follows from results proved in previous sections. □

4. Weak approximation for varieties defined by systems of forms

As an application of the compution of the group of similarity factors, we prove, under some restrictions on the field k, the weak approximation property for k-varieties X defined by the system of forms

$$X \,:\, 0 \neq z = x_1 x_1^J + a_1 y_1 y_1^J = \cdots = x_n x_n^J + a_n y_n y_n^J.$$

Here J is either trivial or a canonical involution of a quaternion division algebra D with center k, $z, x_1, y_1, \ldots, x_n, y_n$ are $2n + 1$ variables, and $a_i \in k^*$. In the second case, if $D = (a, b/k)$ then each x_i, y_i is a tuple of 4 variables $x_i = (x_{1i}, x_{2i}, x_{3i}, x_{4i})$, etc..., and $x_i x_i^J = x_{1i}^2 - a x_{2i}^2 - b x_{3i}^2 + ab x_{4i}^2$ (*the generic norm form*).

This kind of variety over number fields was first considered by Chatêlet, and then by Colliot-Thélène, Sansuc (see e.g. [4, 5]) and others. We have

Proposition 4.1. *Let K, V_0 be as in Proposition* 2.9, *and X the K-variety defined by the following system of forms*:

$$X \,:\, 0 \neq z = x_1^2 + a_1 y_1^2 = \cdots = x_n^2 + a_n y_n^2,$$

where z, x_i, y_i are variables, and $a_i \in K^*$, $1 \geq i \geq n$. Then X satisfies the weak approximation property over K with respect to V_0.

Proof. Let $f_i = \langle 1, a_i \rangle$, $f = f_1 \perp f_2 \ldots \perp f_n$ and $V = P_1 \perp P_2 \perp \cdots \perp P_n$ be the orthogonal decomposition of the quadratic space V associated with f into non-degenerate planes such that the restriction of f to P_i is f_i. Let S_0, S_1, S be as in the proof of Proposition 2.9. We have to show that X has weak approximation with respect to S. We need the following simple lemma.

Lemma 4.2. *Let P be a quadratic space of dimension 2, where in some orthogonal basis of P the associated quadratic form has the diagonal form $\langle 1, a\rangle$. Then for a similarity $g \in \mathrm{GO}(P)$,*

$$g = \begin{bmatrix} x & z \\ y & t \end{bmatrix},$$

g has the similarity factor b if and only if $x^2 + ay^2 = b$ and $t = \pm x, z = \mp ay$.

For $v \in S$ consider a k_v-point

$$(z_v, x_{1v}, \ldots, y_{nv}) \in X(K_v).$$

Since $z_v \in \mathrm{G}(f_v)$, by Proposition 2.9 there is $z \in \mathrm{G}(f)$ such that z is sufficiently close to all z_v for example such that $z_v = zc_v^2$, $c_v \in K_v^*$. Let $g_{i,v} \in \mathrm{GO}(P_{i,v})$, where $P_{i,v} = P_i \otimes K_v$, be a similarity with $z_v = m(g_{i,v})$ and let

$$g_{i,v} = \begin{bmatrix} x_{i,v} & z_{i,v} \\ y_{i,v} & t_{i,v} \end{bmatrix}$$

(see the above lemma). Let $g_v = \mathrm{diag}(g_{1,v}, \ldots, g_{n,v})$, $g \in \mathrm{GO}(V)$ such that $z = m(g)$. It is clear that we have $m(g_v) = m(gc_v)$, hence $g_v = u_v g c_v$, where $u_v \in \mathrm{O}(f_v)$, for all $v \in S$. Since $g_v(P_{i,v}) = P_{i,v}$ for all n, we see that $u_v g(P_{i,v}) = P_{i,v}$ i.e. $g(P_{i,v})$ is isometric to $P_{i,v}$ over K_v, hence zf_i is equivalent to f_i over K_v for all $v \in S$. For $v \notin S$ from the above lemma it follows that $z \in \mathrm{G}(f_{i,v})$. Thus $zf_i \simeq f_i$ over all K_v, hence by the Harder–Milnor Theorem, for any i, $1 \le i \le n$, $zf_i \simeq f_i$ over K. Let $g_i \in \mathrm{GO}(P_i)$ with $z = m(g_i)$. Then $g_{i,v} = u_{i,v} g_i c_v$, where $u_{i,v} \in \mathrm{O}(f_{i,v})$ for any i. Since the groups $\mathrm{O}(f_i)$ and $\mathbf{G}_m$ have the weak approximation property, we can choose $u_i \in \mathrm{O}(f_i)$ (resp. $c \in K^*$) sufficiently close to $u_{i,v}$ (resp. to c_v) for all $v \in S$. Then for

$$u' = \mathrm{diag}(u_1, \ldots, u_n), \; g' = \mathrm{diag}(g_1, \ldots, g_n),$$

the element $u'g'c$ is sufficiently close to g_v for all $v \in S$. Since $(u'g'c)(P_i) = P_i$ for all i, the proposition follows. □

Proposition 4.3. *Let $K = k(t)$ be a rational function field, D a non-trivial quaternion division algebra with center K, and V_0 the set of all valuations of K trivial on k. Assume that for almost all valuations in V_0 the residue class field $\kappa(v)$ of K_v has u-invariant less than or equal to 4. Then the variety X defined by the following system of forms:*

$$X : 0 \neq Z = X_1 X_1^J + a_1 Y_1 Y_1^J = \cdots = X_n X_n^J + a_n Y_n Y_n^J,$$

where $Z, X_1, \ldots, Y_n$ are tuples of independent variables and $a_i \in K^*$, satisfies weak approximation with respect to V_0.

Proof. The proof is similar to the above one. In fact we have to use a unitary analog (see [29, p. 793]) of Lemma 4.2. Let S_0 be any finite set of valuations of V_0, S_1 the set of all valuations $v \in V_0$ such that the u-invariant of the residue class field $\kappa(v)$ of K_v is > 4, and set $S = S_0 \cup S_1$. The method of proof of Proposition 2.9 shows that weak approximation holds for the adjoint group of the K-group $\mathrm{SU}(f)$, where

$$f = x_1 x_1^J + a_1 y_1 y_1^J + \cdots + x_n x_n^J + a_n y_n y_n^J$$

is an hermitian from (of type C) with values in D. In particular, the group $\mathrm{GU}(f)$ also has the weak approximation property. Further the proof goes like before, where we use weak approximation in the group $\mathrm{U}(f)$, the validity of the Harder–Milnor Theorem for hermitian forms of type C (which follows directly from the exact sequence of Witt groups; see [15]). □

The following proposition gives an interesting connection between the weak approximation property of varieties defined by systems of forms and the same property of adjoint groups.

Proposition 4.4. *Let k be a field of characteristic $\neq 2$. Then*

(1) *All adjoint groups of orthogonal groups* $\mathrm{SO}(f)$ *satisfy weak approximation over k if any variety X defined by a system of forms*

$$X : 0 \neq z = x_1^2 + a_1 y_1^2 = \cdots = x_n^2 + a_n y_n^2,$$

where $a_1, \ldots, a_n \in k^*$, satisfies weak approximation over k.

(2) *Conversely, if the weak Hasse principle holds for quadratic forms in two variables over k, and the condition* (S) (*Section* 2) *is fulfilled, then any variety X above satisfies weak approximation over k.*

Proof. (1) It suffices to prove weak approximation for the group $\mathrm{GO}^+(f)$ for any quadratic form f. Let V be the vector space associated with f. Suppose we are given a finite set S of valuations of k, and $g_v \in \mathrm{GO}^+(f)(k_v)$, for $v \in S$. By using [29] and using weak approximation in orthogonal groups, we may assume that to each g_v one can associate an orthogonal decomposition into planes $V_v = P_{1v} \perp \cdots \perp P_{mv}$, each of them invariant under g_v. This corresponds to an orthogonal decomposition $f_v = f_{1v} \perp \cdots \perp f_{mv}$ of f_v into two-dimensional subforms. Using weak approximation in the field k it is not hard to show that there is a k-form f', equivalent to f over k, such that f' has an orthogonal decomposition into two-dimensional subforms $f' = f_1' \perp \cdots \perp f_m'$, where $f_i' \simeq f_{iv}$ over k_v for all $1 \leq i \leq m$, $v \in S$. (To see this we may consider quadratic flags of V, i.e. the flags $0 \subset V_1 \subset V_1 \perp V_2 \subset \cdots \subset V_1 \perp \cdots \perp V_m = V$, where all inclusions are strict inclusions of non-degenerate quadratic subspaces of V. They form an open subset of the flag variety of V, which has the weak approximation property.) Therefore, we may assume that $P_{iv} = P_i \otimes k_v$ for all $v \in S$, $1 \leq i \leq m$, where $V = P_1 \perp \cdots \perp P_m$ is an orthogonal decomposition of V into planes. From Lemma 4.2 the assertion follows.

(2) With the above notation, we need only show that the group $T = \mathrm{GO}^+(f) \cap (\mathrm{GO}^+(f_1) \times \cdots \times \mathrm{GO}^+(f_m))$ has weak approximation. Note that for all i, $T_i = \mathrm{GO}^+(f_i) = \mathbf{G}_m\mathrm{SO}(f_i)$ is a two-dimensional torus. By considering the group T over an algebraic closure of k, it is not hard to see that T is a torus and $T = \mathbf{G}_m L$, where

$$L = \prod_{1 \le i \le m} \mathrm{SO}(f_i),$$

and $\mathbf{G}_m \cap L = \{\pm 1\}$. We have the following exact sequence of k-groups:

$$1 \to L \to T \to \mathbf{G}_m \to 1.$$

From this we can derive the commutative diagram as in the proof of Proposition 2.10. Using the fact that L is k-rational, hence satifies weak approximation over k, with the same method of proof of Proposition 2.10, we conclude that T satisfies weak approximation over k. Using again Lemma 4.2, we see that X satisfies weak approximation. $\square$

5. Integral approximation

In this section we assume that k is a global field. Let G be a k-group, B a subring of k and S a finite set of valuations of k defined by B. We fix once and for all a matrix realisation of G over k, i.e. a faithful k-representation $G \to \mathrm{GL}_n$ for some n. We say that G satisfies *integral approximation over B with respect to S* if $G(B)$ is dense in $\prod_{v \in S} G(O_v)$ in the product of v-adic topologies via the diagonal embedding, where O_v is the ring of v-integers of k and $G(B)$ (resp. $G(O_v)$) is the B- (resp. O_v-) points of G with induced v-adic topology. If it is true for any such S, we say simply that G *satisfies integral approximation over B*. It follows from the definition that if G is a connected k-group for which *strong approximation* (cf. e.g. [9, 12, 22, 23]) holds with respect to a finite set S of valuations of k then G has integral approximation over $O(S)$ with respect to any finite set U of valuations such that $U \cap S = \emptyset$. Denote by O the ring of integers of k and by $O(S)$ the ring of S-integers of k, and set

$$A(S) := \bigcap_{v \in S} (k \cap O_v),$$

where S is a finite set of valuations of k. Let $f : G \to H$ be a morphism of k-groups. We say that S is *admissible for the triple* (f, G, H) if the following two conditions hold:

(1) G, H, f are defined over $O(S)$, i.e. they can be considered as $O(S)$-group schemes and a morphism of $O(S)$-group schemes, respectively;

(2) G, H, f have good reduction modulo v for all $v \notin S$.

It is well-known that for any such triple there exists (non-uniquely) S which is admissible for this triple (cf. e.g. [9]). The following result shows that connected reductive groups over global fields satisfy integral approximation over rings $A(U)$ for any finite set U outside some finite set of valuations. In certain sense it is an analogue of a similar result for weak approximation: For any connected reductive group G over a global field k there is a finite set S of valuations of k such that G satisfies weak approximation with respect to $V \setminus S$, where V is the set of all valuations of k. In the case G is a simply connected almost simple group over a number field k, a similar but stronger result was obtained in [17, Theorem 8.1].

Theorem 5.1. *Let k be a global field and G be a connected k-group with k-split unipotent radical if char. $k > 0$. There is a finite set S of valuations of k such that for any finite set U of valuations of k with $S \cap U = \emptyset$, G satisfies integral approximation over $A(U)$ with respect to U.*

Proof. It is well-known that a k-split unipotent group satisfies strong approximation over k, hence also satisfies integral approximation over $O(S)$ for some finite S. Hence we may assume that G is reductive. Further the proof will be proceeded in several steps.

Step 1: The proposition holds for $G = \mathbf{G}_m$.

We prove that for any finite set U of valuations of k, $\mathbf{G}_m(A(U))$ is dense in $\prod_{v \in U} \mathbf{G}_m(O_v)$. For $v \in U$ let $x_v \in \mathbf{G}_m(O_v) = O_v^*$ (the v-adic units of O_v), and let π_v be a uniformizing element of O_v, N any integer > 0. Then by the Chinese Remainder Theorem there is $x \in O$ such that $x \equiv x_v \pmod{\pi_v^N}$ for all $v \in U$. Since x_v is prime to π_v, x is prime to π_v too. Thus,

$$x \in \bigcap_{v \in U} (O \cap O_v^*) \subset \left(\bigcap_{v \in U} (k \cap O_v) \right)^* = \mathbf{G}_m(A(U)).$$

Step 2: The proposition holds for k-induced tori $G = \mathrm{R}_{K/k}(\mathbf{G}_m)$.

It follows from step 1 by restricting the scalars.

Step 3: The proposition holds for any k-torus G.

Let $1 \to P \to Q \to G \to 1$ be a flasque resolution of G (cf. e.g. [6]), where P is a flasque torus over k and Q is an induced k-torus. Denote by ∞ the set of all archimedean valuations of k. There is a finite set S admissible for both (π, Q, G) and for (i, P, Q), where $i : P \to Q$ is the embedding; we say that S *is admissible for the exact sequence* $1 \to P \to Q \to G \to 1$. We may assume also that $\infty \subset S$ by taking S sufficiently large. Then for any finite set U of valuations of k with $U \cap S = \emptyset$ we have the following commutative diagram with exact rows:

$$\begin{array}{ccccc}
Q(A(U)) & \xrightarrow{\pi} & G(A(U)) & & \\
\downarrow & & \downarrow & & \\
\prod_{v \in U} Q(O_v) & \xrightarrow{\pi'} & \prod_{v \in U} G(O_v) & \longrightarrow & \prod_{v \in U} \mathbf{H}^1(O_v, P)
\end{array}$$

where $\mathbf{H}^1$ stands for étale cohomology. Let $\kappa(v)$ be the residue field of k_v. Then $\mathbf{H}^1(O_v, P) = \mathbf{H}^1(\kappa(v), P) = 0$ by Lang's Theorem hence π' is surjective. Since the assertion is true for Q by previous steps, $Q(A(U))$ is dense in $\prod_{v\in U} Q(O_v)$ hence the assertion also holds for G.

Step 4: The proposition holds for any semisimple k-group G.

Let $\tilde{G}$ be the k-universal covering of G, $\pi : \tilde{G} \to G$ the canonical k-isogeny and $F = \operatorname{Ker} \pi$. It is well-known (by results of Kneser, Platonov, Prasad and Margulis) that there is a finite set S of valuations of k such that $\tilde{G}$ satisfies strong approximation with respect to S (cf. e.g. [23] or [22]). In particular $\tilde{G}$ satisfies integral approximation over $O(S)$. Let $\tilde{T}$ be a maximal k-torus of $\tilde{G}$ and $T = \pi(\tilde{T})$. From previous steps and from what said above, we may choose $S(\supset \infty)$ such that S is admissible for the exact sequences $1 \to F \to \tilde{T} \to T \to 1$ and $1 \to F \to \tilde{G} \to G \to 1$, and such that $\tilde{G}$ satisfies integral approximation over $O(S)$. Now consider the following commutative diagram with exact lines:

$$\begin{array}{ccccccc}
\tilde{T}(A(U)) & \longrightarrow & T(A(U)) & \longrightarrow & \mathrm{H}^1(A(U),F) & & \\
\downarrow & & \downarrow & & \downarrow \gamma & & \\
\prod_{v\in U} \tilde{T}(O_v) & \longrightarrow & \prod_{v\in U} T(O_v) & \longrightarrow & \prod_{v\in U} \mathbf{H}^1(O_v,F) & \longrightarrow & \{0\},
\end{array}$$

where U is a finite set of valuations such that $U \cap S = \emptyset$. Clearly we may also choose S such that the assertion holds for $\tilde{T}$ and T, i.e. $\tilde{T}(A(U))$ (resp. $T(A(U))$) is dense in $\prod_{v\in U} \tilde{T}(O_v)$ (resp. $\prod_{v\in U} T(O_v)$). It is well-known that $\mathbf{H}^1(O_v,F)$ is finite for all $v \in U$. Therefore from above diagram we see that γ is surjective on the image of $T(A(U))$ in $\mathbf{H}^1(A(U)),F)$. By considering similar diagram for the exact sequence $1 \to F \to \tilde{G} \to G \to 1$ and using integral approximation for $\tilde{G}$ and the surjectivity just obtained we can conclude that G also satisfies integral approximation over $A(U)$.

Step 5: The assertion holds for any connected reductive k-group G.

Let $G = TG_1$ where T is a central torus of G and G_1 is semisimple. Let $\pi : T\times G_1 \to G = TG_1$ be the canonical isogeny. Then we may apply the same argument used in step 4. $\square$

Remark. The simplest example $G = \mathbf{G}_m$ shows that in the above theorem, the choice of the ring $A(U)$ is the best possible, in the sense that one cannot take a smaller ring for the integral approximation to hold.

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